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BRAIDED GEOMETRY AND THE INDUCTIVE CONSTRUCTION OF LIE ALGEBRAS AND QUANTUM GROUPS

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Abstract Double-bosonisation associates to a braided group in the category of modules of a quantum group, a new quantum group. We announce the semiclassical version of this inductive construction.

1 Introduction

A question usually overlooked in deformation theory is that of uniformity: we can quantise this or that Poisson manifold, but do our individual quantisations fit together into a coherent 'quantum geometry'? In classical geometry the co-ordinate rings are assumed *uniformly* to be commutative. When we relax this, each object has many 'directions' in which to become non-commutative and we need to know how to pick these in a coherent way.

This problem is addressed by braided geometry, introduced by the author through about 60 papers since 1989. Rather than deforming one algebra at a time, we deform the tensor product itself; we do group theory and geometry in a braided category in place of Vec. Then all mathematical concepts founded in linear algebra are q-deformed uniformly as we switch on the braiding. Braided geometry has its own method of proofs in which algebraic information 'flows' along braid and tangle diagrams like information in a computer, except that under and over crossings of wires are nontrivial braiding operators Ψ [1][2][3][3][4][6][7]. In physical terms, braided geometry is a generalisation of supergeometry with -1 in Bose-Fermi statistics replaced

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by braid statistics (e.g. by q). This is conceptually quite different from the usual quantisation picture where $q = e^{\frac{\hbar}{2}}$. But braided-commutative with respect to some $\otimes_q$ still means non-commutative with respect to the usual $\otimes$, so we generate noncommutative algebras, which we can then 'semiclassicalise' via such an expansion; we do not start with Poisson brackets but rather we generate them, i.e. this is a deeper point of view.

The starting point is the concept of *braided group*[1] or Hopf algebra in a braided category. This means an algebra and coalgebra B in the category for which the coproduct $\underline{\Delta} : B \to B\underline{\otimes}B$ is an algebra homomorphism, where $\underline{\otimes}$ is the *braided tensor product* of algebras[1] in a braided category. In concrete terms, $B\underline{\otimes}B$ has product $(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d$.

The simplest example[4] is the tensor algebra TV on a finite-dimensional vector space V equipped with a braiding $\Psi : V \otimes V \to V \otimes V$. Write $TV = \mathbb{C}\langle x_i \rangle$ and $\Psi(x_i \otimes x_j) = x_b \otimes x_a R^a{}_i{}^b{}_j$ (with summation of indices), where R obeys the Yang-Baxter equation. Then the coproduct has the form[4]:

$$\underline{\Delta} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{r=0}^{m} x_{j_1} \cdots x_{j_r} \otimes x_{j_{r+1}} \cdots x_{j_m} \begin{bmatrix} m \\ r \end{bmatrix}; R \Big]^{j_1 \cdots j_m}_{i_1 \cdots i_m}.$$

In his talk, Rosso[8] mentioned the 'quantum shuffle algebra' but this is just the graded dual of TV. Its product has just the structure of the coproduct of the latter. Writing $y^{i_m \cdots i_1}$ for the dual basis to $x_{i_1} \cdots x_{i_m}$, clearly

$$y^{i_m \cdots i_{r+1}} \cdot y^{i_r \cdots i_1} = \left[\begin{matrix} m \\ r \end{matrix}; R\right]^{i_1 \cdots i_m}_{j_1 \cdots j_m} y^{j_m \cdots j_1}, \quad \underline{\Delta} y^{i_m \cdots i_1} = \sum_{r=0}^{m} y^{i_m \cdots i_{r+1}} \otimes y^{i_r \cdots i_1}.$$

From standard properties[4] of these *braided binomial* matrices $[\begin{smallmatrix} m \\ r \end{smallmatrix}; R]$,

$$\pi : T(V^*) \to (TV)^*, \quad \pi(y^{i_m} y^{i_{m-1}} \cdots y^{i_i}) = [m; R]!^{i_1 \cdots i_m}_{j_1 \cdots j_m} y^{j_m \cdots j_1}$$

is a homomorphism of braided groups, where $[m; R]!$ are the *braided factorial* matrices and $T(V^*) = \mathbb{C}\langle y^i \rangle$. Hence $\mathrm{ev} : T(V^*) \otimes TV \to \mathbb{C}$,

$$\mathrm{ev}(f(y), g(x)) = \pi(f(y))(g(x)) = f(\partial) g(x)|_{x=0} = f(y) g(\overleftarrow{\partial})|_{y=0} \tag{1}$$

is a duality pairing of braided groups. Here ∂ denotes *braided differentiation*

$$\partial^i x_{i_1} \cdots x_{i_m} = x_{j_2} \cdots x_{j_m} [m; R]^{i j_2 \cdots j_m}_{i_1 \cdots i_m}$$

where $[m; R]$ is the *braided integer* matrix. Similarly for $\overleftarrow{\partial}$. These are some rudiments of braided geometry on free algebras[4].

The kernels of ev may be non-zero; quotienting by them gives new braided groups such as the quantum planes $\mathbb{C}_q^n$. Another choice[9] is $R^i{}_j{}^k{}_l = \delta^i{}_j\delta^k{}_l q^{\beta_{jl}}$, where β is a bilinear form. This is the case considered in [8] and Fronsdal's talk[10]. When β comes from a Cartan matrix, Lusztig[12] computed $\ker\pi$ as the q-Serre relations, i.e. $T(V^*)/\ker\pi = \mathrm{image}(\pi) = U_q(n_+)$.

2 Transmutation and Bosonisation; Induction Principle

General theorems about braided groups are the following. We use Sweedler notation $\Delta h = h_{(1)}\otimes h_{(2)}$ for coproducts, S for the antipode and $\mathcal{R} = \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ for Drinfeld's quasitriangular structure (summations understood).

1. Transmutation (SM 1990). Let H be a quantum group (quasitriangular Hopf algebra). Its *transmutation* is the braided group $\underline{H}\in{}_H\mathcal{M}$, the braided category of modules. $\underline{H}$ is H as a module-algebra by Ad, and

$$\underline{\Delta}h = h_{(1)}S\mathcal{R}^{(2)}\otimes \mathrm{Ad}_{\mathcal{R}^{(1)}}(h_{(2)}),\quad \Psi(h\otimes g) = \mathrm{Ad}_{\mathcal{R}^{(2)}}(g)\otimes\mathrm{Ad}_{\mathcal{R}^{(1)}}(h).$$

2. Bosonisation (SM 1991). Let $B\in{}_H\mathcal{M}$ be a braided group with $\underline{\Delta}b = b_{\underline{(1)}}\otimes b_{\underline{(2)}}$ and action $\triangleright$ of H. Its *bosonisation* is the Hopf algebra $B{>\!\!\!\triangleleft} H$ generated by H as a Hopf algebra, B as an algebra, and

$$hb = (h_{(1)}\triangleright b)h_{(2)},\quad \Delta b = b_{\underline{(1)}}\mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}}. \tag{2}$$

3. Biproducts (cf. Radford 1985, SM 1992). Let $B\in{}^F_F\mathcal{M}$, the crossed modules over a Hopf algebra F with bijective S. There is a *biproduct* Hopf algebra $B{>\!\!\!\triangleleft} F$ projecting to F. Every projection to F is of this form.

4. Double-bosonisation (SM 1995). Let $B^\vee$ be dually paired to $B\in{}_H\mathcal{M}$ via $\mathrm{ev}: B\otimes B^\vee\to\mathbb{C}$. There is a quantum group $B{>\!\!\!\triangleleft} H{\triangleright\!\!\!<} B^{\vee\mathrm{op}}$ containing $B{>\!\!\!\triangleleft} H$ and $H{\triangleright\!\!\!<} B^{\vee\mathrm{op}}$ as subHopf algebras, defined by (2) and

$$b_{\underline{(1)}}\mathcal{R}^{(2)}c_{\underline{(1)}}\mathrm{ev}(\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}}, c_{\underline{(2)}}) = \mathrm{ev}(b_{\underline{(1)}},\mathcal{R}^{(2)}\triangleright c_{\underline{(1)}})c_{\underline{(2)}}\mathcal{R}^{(1)}b_{\underline{(2)}}$$
$$hc = (h_{(2)}\triangleright c)h_{(1)},\quad \Delta c = \mathcal{R}^{(2)}\triangleright c_{\underline{(1)}}\otimes c_{\underline{(2)}}\mathcal{R}^{(1)},\quad \mathcal{R}^{\mathrm{new}} = \mathcal{R}\exp^{-1},$$

where $\mathcal{R}^{\mathrm{new}}$ needs a canonical element (coevaluation) $\exp\in B^\vee\otimes B$ for ev.

5. Double-biproducts (SM 1995). Let $B^\vee\in{}^F_F\mathcal{M}$ be dually paired to $B\in{}^F_F\mathcal{M}$ in 3. as in[6]. There is a Hopf algebra $B{>\!\!\!\triangleleft} F{\triangleright\!\!\!<} B^{\vee\mathrm{op}}$. A functor ${}_H\mathcal{M}\to{}^H_H\mathcal{M}$ allows 2. & 4. to be viewed as special cases of 3. & 5.

Bosonisation has been used to construct inhomogeneous Hopf algebras $\mathbb{C}_q^n{>\!\!\!\triangleleft} U_q(\widetilde{su_n})$. The $\tilde{}$ denotes a central extension. On the other hand, double-bosonisation can be iterated to provide a graph of quantum groups, including the standard families of $U_q(g)$ as well as new quantum groups without classical limit. At each node H, the branches are the inequivalent $B \in {}_H\mathcal{M}$. The new node is $B{>\!\!\!\triangleleft} H{\triangleright\!\!\!<} B^{\check{}\mathrm{op}}$. The initial node is the quantum group $\mathbb{C}$. Its central extension is the quantum line $U_q(1)$. Adjoining the braided line $\mathbb{C}_q$ to this yields $U_q(su_2)$. There are several braided groups in the category of $U_q(\widetilde{su_2})$-modules, each yielding a new quantum group. The quantum-braided plane $\mathbb{C}_q^2$ gives us $U_q(su_3)$. There are some technicalities, see [6].

The required quantum-braided planes for induction up the A,B,C,D series $U_q(g)$ are known, while the exceptional series are currently under investigation. As there is surely *some* braided group B in the category of $\widetilde{U_q(e_8)}$-modules, we obtain at least one quantum group $B{>\!\!\!\triangleleft}\widetilde{U_q(e_8)}{\triangleright\!\!\!<} B^{\check{}\mathrm{op}}$ which could be called $U_q(e_9)$! Presumably it does not survive as $q \to 1$. Also, building up $U_q(g)$ inductively by a series of triple products yields automatically a natural *inductive block basis* for it, which becomes a basis when we fix bases for the braided planes B which are adjoined at each stage. For example,

$$U_q(su_n) = \mathbb{C}_q^{n-1}{>\!\!\!\triangleleft}\mathbb{C}_q^{n-2}{>\!\!\!\triangleleft}\cdots{>\!\!\!\triangleleft}\mathbb{C}_q{>\!\!\!\triangleleft} U_q(\beta){\triangleright\!\!\!<}\mathbb{C}_q{\triangleright\!\!\!<}\cdots{\triangleright\!\!\!<}\mathbb{C}_q^{n-2}{\triangleright\!\!\!<}\mathbb{C}_q^{n-1} \tag{3}$$

where the central extensions are collected together as $U_q(\beta) = U(1)^{\otimes n}$ generated by H_i with a quasitriangular structure $\mathcal{R}_\beta = q^{\sum \beta_{ij}^{-1} H_i \otimes H_j}$. Here β is the symmetrised Cartan matrix. This *proves* the PBW theorem for $U_q(g)$ and explicitly constructs $U_q(n_+) = \mathbb{C}_q^{n-1}{>\!\!\!\triangleleft}\mathbb{C}_q^{n-2}\cdots{>\!\!\!\triangleleft}\mathbb{C}_q$. Choosing bases for the $\mathbb{C}_q^i$ gives us a basis for $U_q(n_+)$, as well as all the relations between them (including the q-Serre relations when expressed in terms of the simple roots). The inductive basis is coherent across the graph of quantum groups. Moreover, its restriction to any substring of factors gives a sub-braided or quantum group. In (3), $\mathbb{C}_q^{n-1}{>\!\!\!\triangleleft}\cdots{>\!\!\!\triangleleft}\mathbb{C}_q{>\!\!\!\triangleleft} U_q(\beta) = U_q(b_+)$, $\mathbb{C}_q^2{>\!\!\!\triangleleft}\mathbb{C}_q{>\!\!\!\triangleleft} U_q(\beta){\triangleright\!\!\!<}\mathbb{C}_q = \mathbb{C}_q^2{>\!\!\!\triangleleft} U_q(\widetilde{su_2})$, etc. If one is interested in only half the story, i.e. only in constructing $U_q(b_+)$, one can also do it by iterated biproducts. Thus, $\mathbb{C}_q^n{>\!\!\!\triangleleft} U_q(\widetilde{b_+})$ gives the q-Borel of $U_q(su_{n+1})$.

Double-bosonisation also generalises Lusztig's construction. Any β defines a quantum group $U_q(\beta)$ with generators h_i and $\mathcal{R}_\beta = q^{\sum \beta_{ij} h_i \otimes h_j}$. $B = \mathbb{C}\langle y^i\rangle$, $B^{\check{}} = \mathbb{C}\langle x_i\rangle$, paired by (1), live in the category of $U_q(\beta)$-modules by $h_i{\triangleright} y^j = \delta_{ij} y^j$, $h_i{\triangleright} x_j = -\delta_{ij} x_j$. So $\mathbb{C}\langle y^i\rangle{>\!\!\!\triangleleft} U_q(\beta){\triangleright\!\!\!<}\mathbb{C}\langle x_i\rangle$ is a Hopf

algebra. Quotienting by the kernels of ev we obtain a quantum group $U_q(n_+){\bowtie\!\!\!\triangleleft}U_q(\beta){\triangleright\!\!\!\bowtie}U_q(n_-)$ with $\mathcal{R}=\mathcal{R}_\beta\exp^{-1}$. For generic β (or generic R-matrix in Section 1) $[m;R]$ are invertible and the coevaluation for (1) is

$$\exp=\sum_{m=0}^\infty x_{i_m}\cdots x_{i_1}([m;R]!^{-1})^{i_1\cdots i_m}_{j_1\cdots j_m}y^{j_m}\cdots y^{j_1}\in B^{\check{}\,\mathrm{op}}\otimes B.$$

Otherwise, quotienting by the kernels is nontrivial but we still have ∂, $\overleftarrow{\partial}$ and the *braided exponential* exp is characterised as their eigenfunction[7].

Note that Fronsdal in his talk and [10] considered recursion relations for an ansatz of the form $\mathcal{R}_\beta f(x,y)$ to obey the Yang-Baxter equation, with resulting Hopf algebra being coboundary. By contrast, double-bosonisation already provides a closed expression for $\mathcal{R}$ via a braided-exponential, proves that quotienting by kernels of ev yields a Hopf algebra and proves that it is quasitriangular. [6] has been circulated in October 1995.

3 Braided-Lie Bialgebras and Lie Induction

We now announce a semiclassical concept of braided groups. Let g, $\delta: g\to g\otimes g$, $r\in g\otimes g$ be a quasitriangular Lie bialgebra as per Drinfeld[13]. Let $2r_+=r+\tau(r)$ where τ is transposition. Let $\triangleright$ denote an action of g.

0. A *braided-Lie bialgebra* $b\in{}_g\mathcal{M}$ is a g-covariant Lie algebra and g-covariant Lie coalgebra with cobracket $\underline{\delta}: b\to b\otimes b$ obeying $\forall x,y\in b$,

$$\underline{\delta}([x,y])=\mathrm{ad}_x\delta y-\mathrm{ad}_y\delta x-\psi(x\otimes y);\quad \psi=2r_+(\triangleright\otimes\triangleright)\circ(\mathrm{id}-\tau),$$

i.e., $\mathrm{d}\underline{\delta}=\psi$ where d is the Lie coboundary on $\underline{\delta}\in C^1_{\mathrm{ad}}(b,b\otimes b)$ and $\mathrm{d}\psi\equiv 0$.

1. Let $i: g\to f$ be a map of Lie bialgebras. The *transmutation* of f is a braided-Lie bialgebra $\underline{f}\in{}_g\mathcal{M}$ with Lie algebra f and for all $x\in f$,

$$\underline{\delta}x=\delta x+r^{(1)}\triangleright x\otimes i(r^{(2)})-i(r^{(2)})\otimes r^{(1)}\triangleright x,\quad \triangleright=\mathrm{ad}\circ i.$$

In particular, g has a braided version $\underline{g}\in{}_g\mathcal{M}$ by ad, the same bracket, and

$$\underline{\delta}x=2{r_+}^{(1)}\otimes[x,{r_+}^{(2)}].\tag{4}$$

2. Let $b\in{}_g\mathcal{M}$ be a braided-Lie bialgebra. Its *bosonisation* is the Lie bialgebra $b{>\!\!\!\triangleleft}g$ with g as sub-Lie bialgebra, b as sub-Lie algebra and

$$[\xi,x]=\xi\triangleright x,\quad \delta x=\underline{\delta}x+r^{(2)}\otimes r^{(1)}\triangleright x-r^{(1)}\triangleright x\otimes r^{(2)},\quad \forall\xi\in g,\ x\in b.\tag{5}$$

3. Let f be a Lie bialgebra and ${}^f_f\mathcal{M}$ its category of Lie crossed modules (=modules of the Drinfeld double $D(f)$.) Objects b are simultaneously f-modules $\triangleright$ and f-comodules $\beta : b \to f \otimes b$ obeying $\forall \xi \in f, x \in b$,

$$\beta(\xi \triangleright x) = ([\xi,] \otimes \mathrm{id} + \mathrm{id} \otimes \xi \triangleright)\beta(x) + (\delta\xi)\triangleright x.$$

Writing $\beta(x) = x^{(\bar{1})} \otimes x^{(\bar{2})}$, the infinitesimal braiding in this category is

$$\psi(x \otimes y) = y^{(\bar{1})}\triangleright x \otimes y^{(\bar{2})} - x^{(\bar{1})}\triangleright y \otimes x^{(\bar{2})} - y^{(\bar{2})} \otimes y^{(\bar{1})}\triangleright x + x^{(\bar{2})} \otimes x^{(\bar{1})}\triangleright y.$$

Let $b \in {}^f_f\mathcal{M}$ be a braided-Lie bialgebra. The *bisum* Lie bialgebra $b{>\!\!\!\triangleleft\kern-.33em\cdot\,} g$ has semidirect Lie bracket/cobracket and projects onto f. Any Lie bialgebra projecting onto f is of this form. A functor ${}_g\mathcal{M} \to {}^g_g\mathcal{M}$ relates 2. & 3.

4. Let $b^{\check{}} \in {}_g\mathcal{M}$ be a braided-Lie bialgebra dually paired with b by invariant $\mathrm{ev} : b \otimes b^{\check{}} \to \mathbb{C}$. Its *double-bosonisation* is the Lie bialgebra $b{>\!\!\!\triangleleft} g {\triangleright\!\!\!<} b^{\check{}\mathrm{op}}$ with g as sub-Lie bialgebra, $b, b^{\check{}\mathrm{op}}$ sub-Lie algebras, (5) and

$$[\xi, \phi] = \xi\triangleright\phi, \quad [x, \phi] = \mathrm{ev}(x_{\underline{(1)}}, \phi)x_{\underline{(2)}} + \mathrm{ev}(x, \phi_{\underline{(1)}})\phi_{\underline{(2)}} + 2r_+{}^{(1)}\mathrm{ev}(x, r_+{}^{(2)}\triangleright\phi)$$
$$\delta\phi = \underline{\delta}\phi + r^{(2)}\triangleright\phi \otimes r^{(1)} - r^{(1)} \otimes r^{(2)}\triangleright\phi, \quad r^{\mathrm{new}} = r - \sum_a f^a \otimes e_a,$$

$\forall x \in b, \xi \in g$ and $\phi \in b^{\check{}}$. Here $\underline{\delta}x = x_{\underline{(1)}} \otimes x_{\underline{(2)}}$, etc., and r^{new} assumes that ev has a coevaluation, i.e. if $\{e_a\}$ is a basis of b then $\{f^a\}$ is dual w.r.t. ev.

Double bosonisation provides an inductive construction for quasitriangular Lie bialgebras, preserving factorisability (nondegeneracy of r_+). It is a co-ordinate free version of the idea of adjoining a node to a Dynkin diagram (adjoining a simple root vector in the Cartan-Weyl basis). Moreover, building up g iteratively like this also builds up the quasitriangular structure r. The braided-Lie bialgebra used in the induction could be trivial:

Proposition. Let g be a semisimple factorisable (s.s.f) Lie bialgebra and b a faithful isotypical representation such that $\Lambda^2 b$ is isotypical. Then b with zero bracket and zero cobracket is a braided-Lie bialgebra in ${}_{\tilde{g}}\mathcal{M}$, and $b{>\!\!\!\triangleleft}\tilde{g}{\triangleright\!\!\!<} b^{\check{}\mathrm{op}}$ is another s.s.f. Lie bialgebra. Here $\tilde{g}$ is a central extension.

The induction also works at the simple strictly quasitriangular level (with b irreducible). For example, the 2-dimensional and 3-dimensional representations of su_2 have the required property

(ensuring $\psi \propto (\mathrm{id} - \tau)$ in ${}_g\mathcal{M}$). They give su_3 and so_5, taking us up the A and B series respectively.

Finally, just as Lie bialgebras extend to Poisson-Lie groups, so braided-Lie bialgebra structures generally extend to the associated Lie group of g. The resulting Poisson bracket does not, however, respect the group product in the usual way but rather up to a 'braiding' obtained from ψ.

Example. The transmutation (4) of the Drinfeld-Sklyanin (or other factorisable) Lie cobracket on semisimple g is the Kirillov-Kostant Lie cobracket. Moreover, this *Kirillov-Kostant braided-Lie bialgebra* extends, in principle, to a braided-Poisson Lie group. Details are to appear elsewhere.

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